Wigner Functions in Curved Spacetimes and Deformation Quantisation of Constrained Systems

Frank Antonsen
Niels Bohr Institute

We first generalise the standard Wigner function to Dirac fermions in curved spacetimes. Secondly, we turn to the Moyal quantisation of systems with constraints. Gravity is used as an example.

1 Curved Spacetime Wigner Function

Probably the greatest unsolved problem of modern theoretical physics is the interplay between quantum theory and gravity (as described by general relativity).
The simplest "subproblem" is the study of quantum systems in a given, fixed classical gravitational field, i.e., in a given curved background.
The Wigner function in flat space is

$$W(q,p) := \int \bar{\psi}(q-\frac{1}{2}y)\psi(q+\frac{1}{2}y)e^{-iyp}\frac{d^n y}{(2\pi)^n} \tag{1}$$

in n dimensions. The problem in curved spacetimes is the definition of $q \pm \frac{1}{2}y$, since this in general doesn't make sense in non-flat spaces. In [1] I showed how to generalise W to a Dirac spinor in a given curved background. The Dirac equation for ψ then induces the following equation for W

$$\left[m + \gamma^\mu\left(e^a_\mu p_a + \frac{1}{2}i\nabla_\mu\right)\right]\hat{W} = -\frac{1}{2}\kappa\gamma^a\hat{X}_a\hat{W} \tag{2}$$

where e^a_μ is a vierbein and where $\hat{X}_a$ is an infinite order differential operator involving the curvature tensor.
This allows one to find non-perturbative expressions for various macroscopic quantities (i.e., quantum magneto-hydrodynamics). With this, a phasespace interpretation of e.g. the conformal anomaly can be obtained, see [1] for further details.

2 Hamiltonian Systems with Constraints

Gravitation itself is described by a set of constraints. The Hamiltonian itself is nothing but a linear combination of constraints. This is in contrast to the situation for other gauge fields (Maxwell or Yang-Mills), implying that we

cannot simply import known results concerning gauge fields to gravitational degrees of freedom.
Instead what we will do, is to perform a Moyal (or deformation) quantisation. I.e., replace the classical Poisson brackets $\{\cdot,\cdot\}_{\rm PB}$ by Moyal brackets $[\cdot,\cdot]_M$.

$$\begin{aligned}[f,g]_M &= i\hbar\{f,g\}_{\rm PB}+O(\hbar^2) &(3)\\ &= 2if\sin\left(\frac{1}{2}\hbar\{\cdot,\cdot\}_{\rm PB}\right)g &(4)\\ &= f*g-g*f &(5)\end{aligned}$$

Consequently, if we have a set of classical constraints $\phi_a(q,p)$, satisfying $\{\phi_a,\phi_b\}_{\rm PB}=c^c_{ab}\phi_c$ (i.e., being first class) we want to find a corresponding set of quantum constraints $\Phi_a=...\hbar^{-1}\Phi^{(-1)}_a+\phi_a+\hbar\Phi^{(1)}_a+...$ satisfying

$$[\Phi_a,\Phi_b]_M=i\hbar c^c_{ab}\Phi_c \tag{6}$$

It has been proven in [2] that classical second class constraints (i.e., constraints satisfying $\{\phi_a,\phi_b\}_{\rm PB}\neq c^c_{ab}\phi_c$) can be turned into quantum first class constraints by allowing a $\hbar^{-1}$ term in Φ_a. If we cannot take $\Phi_a=\phi_a$ then we say we have an anomaly. It was also proven in [2] how such anomalies could in certain circumstances be "lifted", i.e., quantum constraints Φ_a did exist satisfying the appropriate quantum (Moyal) constraint algebra. This is the case if the anomaly is merely a central extension.
Classically, physical states are defined by the requirement $\forall a\ :\ \phi_a=0$. In the standard Dirac quantisation scheme, this is interpreted as the condition $\forall a\ :\ \hat{\phi}_a|\psi\rangle=0$ picking out physical states $|\psi\rangle$. In a deformation quantisation we must instead introduce the BRST-like condition

$$[\Phi_a,W]^+_M:=2\Phi_a\cos\left(\frac{1}{2}\hbar\{\cdot,\cdot\}_{\rm PB}\right)W=\Phi_a*W+W*\Phi_a=0 \tag{7}$$

on the would-be physical Wigner functions W. In general, this will be an infinite order differential equation.

3 Gravity

It turns out, that gravity in both the ADM approach (i.e., where the phasespace is parametrised by the set of spatial 3-metrics and their conjugate momenta) and in the Ashtekar variables approach (where one has a densitised dreibein E^i_a - a kind of electrical field - whose conjugate is a complex $SU(2)$ connection A^a_i - the analogue of the Yang-Mills connection) is anomalous. However, in

the Ashtekar variables this anomaly is exceedingly simple

$$[\mathcal{H}(x), \mathcal{D}_i(x')]_M = i\hbar\{\mathcal{H}(x), \mathcal{D}_i(x')\}_{\mathrm{PB}} - 9i\hbar^3\delta_{,i}(x,x') \tag{8}$$

being merely a central extension (a Schwinger term). Here $\mathcal{H} = F^a_{ij}E^i_bE^j_c\varepsilon_{bc}{}^a$ is the Hamiltonian constraint (in the obvious notation with F^a_{ij} the field strength tensor of A^a_i), and $\mathcal{D}_i = F^a_{ij}E^j_a$ the diffeomorphism one. The above Moyal bracket is the only anomalous one.
Furthermore, the equations for physical states become finite order, whereas they become infinite order in the ADM-approach.

$$\begin{aligned}
0 = [\mathcal{H}, W]^+_M &= 2\mathcal{H}W - \frac{1}{2}\hbar^2 \left(E^k_b E^l_v \epsilon_a{}^{bc}\epsilon^a{}_{ef}\frac{\delta^2 W}{\delta E^k_e \delta E^l_f} - \right. \\
&\quad 2\epsilon_a{}^{bc}\left(-\delta^a_e(\delta^k_i\partial_j - \delta^k_j\partial_i) + \epsilon^a{}_{pq}(\delta^p_e\delta^k_i A^q_j + \delta^q_e\delta^k_j A^p_i)\right) \times \\
&\quad (\delta^i_l\delta^b_f E^j_c + \delta^j_l\delta^c_f E^i_b)\frac{\delta^2 W}{\delta E^k_e \delta A^f_l} + \\
&\quad \left. \epsilon_a{}^{bc}F^a_{ij}\frac{\delta^2 W}{\delta A^b_i \delta A^c_j}\right) + \frac{5}{4}\hbar^4\epsilon_{bc}{}^a\epsilon_a{}^{ef}\frac{\delta^4 W}{\delta E^k_e \delta E^l_f \delta A^e_k \delta A^f_l}
\end{aligned} \tag{9}$$

$$\begin{aligned}
0 = [\mathcal{D}_i, W]^+_M &= 2\mathcal{D}_i W - \frac{1}{2}\hbar^2 \left(\epsilon^a{}_{ef}E^j_a\frac{\delta^2 W}{\delta E^i_e \delta E^j_f} - \right. \\
&\quad 2\left(-\delta^a_e(\delta^k_i\partial_j - \delta^k_j\delta_i) + \epsilon^a{}_{mn}(\delta^m_e\delta^k_i A^m_j + \delta^n_e\delta^k_j A^m_i)\right)\frac{\delta^2 W}{\delta E^k_e \delta A^a_j}
\end{aligned} \tag{10}$$

$$0 = [\mathcal{G}_a, W]^+_M = 2\mathcal{G}_a W + \frac{1}{4}i\hbar^2\delta^j_k\epsilon^c{}_{ab}\frac{\delta^2 W}{\delta A^c_j \delta A^k_b} \tag{11}$$

with $\mathcal{G}_a = D_iE^i_a$ the Gauss constraint.

References

1. F. Antonsen, Phys. Rev D**56** (1997) 920, hep-th/9701182.
2. F. Antonsen, gr-qc/9710021 (submitted).